New Hierarchies of Derivative nonlinear Schrödinger Type Equation

Jingsong He

Department of Mathematics
Ningbo University, Ningbo, Zhejiang

Joint with Zhiwei Wu (Ningbo University), Yongshuai Zhang (USTC)

presentation at
International workshop on integrable systems, NTU, Taipei, Taiwan
19th-21st October, 2015

Outline for Lecture

- Introduction
- Lie algebra splitting and soliton hierarchy
- Construction of hierarchies of DNLS type equations
- Algebraic structure of nonlocal NLS and DNLS type hierarchies
- PT symmetric solution for nonlocal NLS equation
- Comments and future work

Physics background

- The nonlinear Schrödinger (NLS) equation.
  1. $q_t = \frac{i}{2}(q_{xx} + 2|q|^2 q)$ (focusing NLS).
  2. optical structures, Bose-Einstein condensates, $\cdots$
- The derivative nonlinear Schrödinger (DNLS) equation.
  1. $q_t = \frac{1}{2}(q_{xx} i \mp (|q|^2 q)_x)$.
  2. plasma, optical waveguides with the self-steepening effect, rouge wave, $\cdots$
- Nonlocal reduction ($\mathcal{PT}$-symmetry) of NLS.
  1. Ablowitz-Musslimani (2013): $r(x) = -\bar{q}(-x)$ in AKNS.
  2. Lax pair, inverse scattering method, Hamiltonian formulation.
  3. Experimental evidences: optical waveguides, photonics lattices, microresonators, $\cdots$

DNLSI, II, III equations

NLS: R.Y. Chiao, E. Garmire and C.H. Townes, PRL13(1964)479; V.E. Zakharov, J.Appl.Mech. and Tech.Phys. 9(1968) 190

$$q_t = \frac{i}{2}(q_{xx} \pm 2|q|^2 q). \tag{1}$$

DNLSI: Kaup-Newell (JMP 19(1978)798),

$$q_t = \frac{1}{2}(q_{xx} i \mp (|q|^2 q)_x), \tag{2}$$

DNLSII: Chen-Lee-Liu (Phys.Scr.20(1979)490)

$$q_t = \frac{1}{2}(q_{xx} i \mp |q|^2 q_x), \tag{3}$$

DNLSIII: Gerdjikov-Ivanov (Bulg. J. Phys. 10(1983)130), Jyh-Hao Lee(Transactions of American Mathematical Society 314(1989)107-118)

$$q_t = \frac{i}{2} q_{xx} \pm \frac{1}{2} q^2 \bar{q}_x + \frac{i}{4} |q|^4 q. \tag{4}$$

Lie algebra splitting

Integrable systems can be derived from Lie algebra splitting:

- Ablowitz-Kaup-Newell-Segur (1974).
- Jimbo-Miwa (1983).
- Drinfel'd-Sokolov (1984).
- Segal-Wilson (1985).
- Terng-Uhlenbeck (2000).

Notation:

- G: compact Lie group, $\mathcal{G}$: Lie algebra of G.
- $L(G)$: the group of sooth loops from S^1 to G, $\mathcal{L}(\mathcal{G})$: the Lie algebra of $L(G)$.

$$\mathcal{L}(\mathcal{G}) = \left\{ \sum_{i \le n_0} A_i \lambda^i \mid A_i \in \mathcal{G}, n_0 \in \mathbb{Z} \right\}.$$

G-hierarchy

- Splitting:

$$\begin{cases} L(G)_+, L(G)_- \subset L(G), \quad L(G)_+ \cap L(G)_- = \{e\}, \\ \mathcal{L}(\mathcal{G}) = \mathcal{L}(\mathcal{G})_+ \oplus \mathcal{L}(\mathcal{G})_- (\text{as direct sum of linear subspaces}). \end{cases}$$

- Vacuum sequence: a sequence of commuting elements in $\mathcal{L}(\mathcal{G})_+$: $\mathcal{J} = \{J_1, J_2, \cdots\}$.
- Phase space: $\mathcal{M} = \pi_+(g_- J_1 g_-^{-1}), \quad g_- \in L(G)_-$.

Theorem (Terng-Uhlenbeck, 2000)

Given $\xi : \mathbb{R} \to \mathcal{M}$, there exists a unique $Q_j(\xi) \in \mathcal{L}(\mathcal{G})$ such that:

$$\begin{cases} [\partial_x + \xi, Q_j(\xi)] = 0, \\ Q_j(J_1) = J_j, \quad Q_j(\xi) \text{ is conjugate to } J_j. \end{cases}$$

The *j-th flow in the G-hierarchy* :

$$\xi_{t_j} = [\partial_x + \xi, (Q_j(\xi))_+].$$

(G, σ)-hierarchy

Let σ be an involution of G such that the induced involution $\mathrm{d}_e\sigma$ on $\mathcal{G}$ is complex linear (still using σ to denote $\mathrm{d}_e\sigma$).

$$\mathcal{L}_\sigma(\mathcal{G}) = \{A(\lambda) \in \mathcal{L}(\mathcal{G}) \,|\, \sigma(A(-\lambda)) = A(\lambda)\}.$$

Splitting: $\mathcal{L}_\sigma(\mathcal{G}) = \mathcal{L}_\sigma(\mathcal{G})_+ \oplus \mathcal{L}_\sigma(\mathcal{G})_-$.
Vacuum sequence: $\mathcal{J} = \{J_1, J_2, \cdots\} \in \mathcal{L}_\sigma(\mathcal{G})_+$.
Phase space: $\mathcal{M} = \pi_+(g_- J_1 g_-^{-1}), \quad g_- \in L_\sigma(G)_-$.

Theorem (Terng-Uhlenbeck, 2000)

Given $\xi \in C^\infty(\mathbb{R}, \mathcal{M})$, there exists a unique $Q_j(\xi) \in \mathcal{L}_\sigma(\mathcal{G})$ for any $j \geq 1$ such that

$$\begin{cases} [\partial_x + \xi, Q_j(\xi)] = 0, \\ Q_j(J_1) = J_j, \quad Q_j(\xi) \text{ is conjugate to } J_j. \end{cases}$$

The *j-th flow in the (G, σ)-hierarchy* is

$$[\partial_x + \xi, \partial_{t_j} + (Q_j(\xi))_+] = 0.$$

$SU(2)$-hierarchy and the defocusing NLS equation

- $\mathcal{L}(su(2)) = \left\{ A(\lambda) = \sum_i A_i \lambda^i \mid A_i \in su(2), A(\lambda) = -\overline{A(\bar{\lambda})}^t \right\},$

$$\begin{cases} \mathcal{L}_+(su(2)) = \{\sum_{i\geq 0} A_i \lambda^i \mid A_i \in su(2)\}, \\ \mathcal{L}_-(su(2)) = \{\sum_{i<0} A_i \lambda^i \mid A_i \in su(2)\}. \end{cases}$$

- $a = \mathrm{diag}(i. - i)$, $J_1 = a\lambda$, and $\mathcal{J} = \{a\lambda^i \mid i \geq 1\}$.
- $\xi = J_1 + u = a\lambda + \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}, \quad q \in C^\infty(\mathbb{R}, \mathbb{C})$.
- Solve
  $Q(u, \lambda) = a\lambda + Q_0 + Q_{-1}\lambda^{-1} + Q_{-2}\lambda^{-2} \cdots \in \mathcal{L}(su(2))$:

$$Q_0 = u, \quad Q_{-1} = \frac{i}{2}\begin{pmatrix} -|q|^2 & q_x \\ \bar{q}_x & |q|^2 \end{pmatrix},$$

$$Q_{-2} = \frac{1}{4}\begin{pmatrix} q_x\bar{q} - q\bar{q}_x & -q_{xx} - 2|q|^2 q \\ \bar{q}_{xx} + 2|q|^2\bar{q} & q\bar{q}_x - q_x\bar{q} \end{pmatrix}.$$

- The second flow is the focusing NLS equation.

$U(1,1)$-hierarchy and defocusing NLS equation

Let $U(1,1)$ be the subgroup of $SL(2,\mathbb{C})$ preserving the bilinear form in $\mathbb{C}^2$:

$$\langle X, Y\rangle = \bar{X}^t I_{1,1} Y, \quad I_{1,1} = \mathrm{diag}(1,-1), \quad X, Y \in \mathbb{C}^2.$$

Let $u(1,1)$ be the Lie algebra for $U(1,1)$:

$$u(1,1) = \{g \in sl(2,\mathbb{C}) \mid \bar{g}^t I_{1,1} + I_{1,1} g = 0\} = \left\{ \begin{pmatrix} \alpha i & \beta \\ \bar{\beta} & -\alpha i \end{pmatrix} \mid \alpha \in \mathbb{R}, \beta \in \mathbb{C} \right\}.$$

- Splitting: $\begin{cases} \mathcal{L}_+(u(1,1)) = \{\sum_{i\geq 0} A_i \lambda^i \mid A_i \in u(1,1)\}, \\ \mathcal{L}_-(u(1,1)) = \{\sum_{i<0} A_i \lambda^i \mid A_i \in u(1,1)\}. \end{cases}$
- $\mathcal{J} = \{a\lambda^i \mid i \geq 1\}$.
- $\xi = a\lambda + u = a\lambda + \begin{pmatrix} 0 & q \\ \bar{q} & 0 \end{pmatrix}, \quad q \in C^\infty(\mathbb{R}, \mathbb{C})$.
- The second flow is the defocusing NLS equation:

$$q_t = \frac{i}{2}(q_{xx} - 2|q|^2 q). \tag{5}$$

The Kaup-Newell (KN) system

Scheme of the construction:

- Involution σ on $sl(2,\mathbb{C})$:

$$\sigma(A) = I_{1,1} A I_{1,1}^{-1}, \quad I_{1,1} = \mathrm{diag}(1,-1).$$

- $$\begin{cases} \mathcal{L}_\sigma(sl(2,\mathbb{C})) = \{A(\lambda) = \sum_{i\le n_0} A_i\lambda^i \mid I_{1,1}A(-\lambda)I_{1,1} = A(\lambda)\}, \\ \mathcal{L}_\sigma(sl(2,\mathbb{C}))_+ = \{\sum_{i\ge 1} A_i\lambda^i \in \mathcal{L}_\sigma(sl(2,\mathbb{C}))\}, \\ \mathcal{L}_\sigma(sl(2,\mathbb{C}))_- = \{\sum_{i\le 0} A_i\lambda^i \in \mathcal{L}_\sigma(sl(2,\mathbb{C}))\}. \end{cases}$$

- $$\mathcal{K} = \mathrm{diag}(\alpha,-\alpha), \quad \mathcal{P} = \begin{pmatrix} 0 & \beta \\ \eta & 0 \end{pmatrix}, \quad \alpha,\beta,\eta \in \mathbb{C}.$$

- $A(\lambda) = \sum_i A_i\lambda^i \in \mathcal{L}_\sigma(sl(2,\mathbb{C}))$ if and only if

$$\begin{cases} A_i \in \mathcal{K}, & i \quad \text{even}, \\ A_i \in \mathcal{P}, & i \quad \text{odd}. \end{cases}$$

- Vacuum sequence: $\mathcal{J} = \{a\lambda^{2j} \mid j \ge 1\}$.

KN system (continue)

- Given $u = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} \in \mathcal{P}$, solve
  $Q(u,\lambda) = a\lambda^2 + Q_1\lambda + Q_0 + Q_{-1}\lambda^{-1} + \cdots \in \mathcal{L}_\sigma(sl(2,\mathbb{C}))$
  from
  $$\begin{cases} [\partial_x + a\lambda^2 + u\lambda, Q(u,\lambda)] = 0, \\ Q(u,\lambda)^2 = -\lambda^4. \end{cases}$$

Example (Explicit formula for first several coefficients)

$$Q_1 = u, \quad Q_0 = \frac{i}{2}\begin{pmatrix} qr & 0 \\ 0 & -qr \end{pmatrix}, \quad Q_{-1} = \frac{1}{2}\begin{pmatrix} 0 & q_x i + q^2 r \\ -r_x i + qr^2 & \end{pmatrix},$$

$$Q_{-2} = \frac{1}{8}\begin{pmatrix} 2(qr_x - q_x r) + 3q^2r^2 i & 0 \\ 0 & 2(q_x r - qr_x) - 3q^2r^2 i \end{pmatrix},$$

$$Q_{-3} = -\frac{1}{8}\begin{pmatrix} 0 & 2q_{xx} - 6qrq_x i - 3q^3r^2 \\ 2r_{xx} + qrr_x i - 3q^2r^3 & 0 \end{pmatrix}.$$

The second and third flows and DNLSI

The second flow is the KN system,

$$\begin{cases} q_t = \frac{1}{2}(q_{xx}i + (q^2r)_x), \\ r_t = \frac{1}{2}(-r_{xx}i + (qr^2)_x). \end{cases}$$

The third flow is:

$$\begin{cases} q_t = -\frac{1}{4}q_{xxx} + \frac{3i}{4}(qrq_x)_x + \frac{3}{8}(q^3r^2)_x, \\ r_t = -\frac{1}{4}r_{xxx} - \frac{3i}{4}(qrr_x)_x + \frac{3}{8}(q^2r^3)_x. \end{cases}$$

Example

Let $G = SU(2)$, the second flow is the DNLSI equation (2) $(r = -\bar{q})$.

Theorem (He-Wu, 2015)

Equations belonging to the KN system admits the constraint $r = \pm\bar{q}$.

Derivative nonlinear Schrödinger equation II

- $\mathcal{B}:\mathcal{K}\to\mathcal{K}$ linear: $\mathcal{B}(A)=(1-2\alpha i)A, \alpha\in\mathbb{C}$.
- Projection: $(\sum_i A_i\lambda^i)_+ = \sum_{i\geq 1} A_i\lambda^i + A_0 - \mathcal{B}(A_0)$.
- Vacuum sequence: $\mathcal{J}=\{a\lambda^2, a\lambda^4,\cdots\}$
- $\mathcal{M}=a\lambda^2+u\lambda+P_0=a\lambda^2+\begin{pmatrix}0 & q\\ r & 0\end{pmatrix}\lambda-\alpha\begin{pmatrix}qr & 0\\ 0 & -qr\end{pmatrix}$.
- Let $Q(u,\lambda)=a\lambda^2+Q_1\lambda+Q_0+\cdots$ be the unique element in $\mathcal{L}_\sigma(sl(2,\mathbb{C}))$, such that:

$$\begin{cases}[\partial_x+a\lambda^2+u\lambda+P_0, Q(u,\lambda)]=0,\\ Q(u,\lambda)^2=-\lambda^4.\end{cases}$$

The *j-th flow in the $(SL(2,\mathbb{C}),\sigma)$-hierarchy of twisted by $\mathcal{B}$* is

$$u_{t_j}=(Q_{3-2j})_x+[P_0,Q_{3-2j}]+[u,Q_{2-2j}-\mathcal{B}(Q_{2-2j})]. \tag{6}$$

Second flow

$$Q_1 = u, \quad Q_0 = \frac{i}{2}\begin{pmatrix} qr & 0 \\ 0 & -qr \end{pmatrix},$$

$$Q_{-1} = \frac{1}{2}\begin{pmatrix} 0 & q_x i - (2i\alpha - 1)q^2 r \\ -r_x i - (2i\alpha - 1)qr^2 & \end{pmatrix},$$

$$Q_{-2} = \frac{1}{4}\begin{pmatrix} qr_x - q_x r + (4\alpha + \frac{3}{2}i)q^2r^2 & 0 \\ 0 & q_x r - qr_x - (4\alpha + \frac{3}{2}i)q^2r^2 \end{pmatrix}.$$

The second flow is:

$$\begin{cases} q_t = \frac{i}{2}q_{xx} - (i\alpha - \frac{1}{2})(q^2r)_x - i\alpha q^2 r_x + (\frac{1}{2}\alpha - 2i\alpha^2)q^3r^2, \\ r_t = -\frac{1}{2}ir_{xx} - (i\alpha - \frac{1}{2})(qr^2)_x - i\alpha q_x r^2 + (2i\alpha^2 - \frac{1}{2}\alpha)q^2r^3. \end{cases} \tag{7}$$

Consider the real form $SU(2)$ of $SL(2,\mathbb{C})$, then (7) becomes:

$$q_t = \frac{i}{2}q_{xx} + (2\alpha i - 1)|q|^2 q_x + (2\alpha i - \frac{1}{2})q^2\bar{q}_x + (\frac{1}{2}\alpha - 2\alpha^2 i)|q|^4 q, \tag{8}$$

where α is pure imaginary.

Example

- When $\alpha = 0$, (8) is the DNLSI equation.
- When $\alpha = -\frac{i}{4}$, (8) is the DNLSII equation.
- When $\alpha = -\frac{i}{2}$, (8) is the DNLSIII equation.

Remark: The other equations in (2), (3) and (4) are derived by choosing $G = U(1,1)$.

Theorem (He-Wu, 2015)

The j-th flows in generalized KN hierarchies admit the constraints $r = \pm\bar{q}$ for each $j \in \mathbb{N}$.

Nonlocal DNLS-type equations

Theorem (He-Wu, 2015)

The even flows in the generalized KN hierarchies admits nonlocal constraints of the type $r(x,t) = \pm i\bar{q}(-x,t)$.

Main idea of the proof:
We prove this theorem by finding the algebra structure for each case then deriving the flows from Lie algebra splitting with certain automorphisms.

Nonlocal DNLSI equation

- $\xi : C^\infty(\mathbb{R}, sl(2,\mathbb{C})) \to C^\infty(\mathbb{R}, sl(2,\mathbb{C}))$:

$$\xi(A)(x) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \bar{A}(-x) \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

- $\xi(A)$ induces an automorphism $\tilde{\xi}$ on $\mathcal{L}_\sigma(sl(2,\mathbb{C}))$:

$$\tilde{\xi}(f(x,t,\lambda)) = \sum_j \xi(f_j(-x,t))(i\lambda)^j. \tag{9}$$

- Let $\mathcal{G}_j$ be the eigenspace of ξ in $C^\infty(\mathbb{R}, sl(2,\mathbb{C}))$ with respect to eigenvalue i^j, $0 \le j \le 3$.
- $\mathcal{L}_{\sigma,\xi}(sl(2,\mathbb{C}))$ the set of fixed points of $\xi(A)$ on $\mathcal{L}_\sigma(sl(2,\mathbb{C}))$.

$f(x,t,\lambda) = \sum_j \xi(f_j(x,t))(i\lambda)^j \in \mathcal{L}_{\sigma,\xi}(sl(2,\mathbb{C}))$ if and only if

$$f_{4k+j}(x,t) \in \mathcal{G}_j, \quad k \in \mathbb{Z}, \quad 0 \le j \le 4.$$

Nonlocal DNLSI equation

- $Q(u,\lambda) = a\lambda^2 + Q_1\lambda + Q_0 + Q_{-1}\lambda^{-1} + \cdots \in \mathcal{L}_{\sigma,\xi}(sl(2,\mathbb{C}))$:

$$\begin{cases} [\partial_x + a\lambda^2 + u\lambda, Q(u,\lambda)] = 0, \\ Q(u,\lambda)^2 = -\lambda^4. \end{cases}$$

Example

$$Q_1 = u, \quad Q_0 = \frac{1}{2}\begin{pmatrix} q(x,t)\bar{q}(-x,t) & 0 \\ 0 & -q(x,t)\bar{q}(-x,t) \end{pmatrix},$$

$$Q_{-1} = \frac{1}{2}\begin{pmatrix} 0 & i(q_x(x,t) - q^2(x,t)\bar{q}(-x,t)) \\ \bar{q}_x(-x,t) - q(x,t)\bar{q}^2(-x,t) & 0 \end{pmatrix}.$$

Therefore, the second flow is

$$q_t(x,t) = \frac{i}{2}q_{xx}(x,t) - 2iq(x,t)\bar{q}(-x,t)q_x(x,t) + \frac{i}{2}q^2(x,t)\bar{q}_x(-x,t).$$

Three types of nonlocal DNLS equations

The nonlocal DNLSI equation

$$q_t(x,t) = \frac{i}{2} q_{xx}(x,t) \pm 2iq(x,t)\bar{q}(-x,t)q_x(x,t) \mp \frac{i}{2} q^2(x,t)\bar{q}_x(-x,t).$$

The nonlocal DNLSII equation

$$q_t(x,t) = \frac{i}{2} q_{xx}(x,t) \pm \frac{i}{2} q(x,t)\bar{q}(-x,t)q_x(x,t).$$

The nonlocal DNLSIII equation

$$q_t(x,t) = \frac{i}{2} q_{xx}(x,t) \pm \frac{i}{2} q^2(x,t)\bar{q}_x(-x,t) - \frac{i}{4} q^3(x,t)\bar{q}^2(-x,t).$$

PT symmetric solution for nonlocal NLS equation

The nonlocal NLS equation

$$q_t - \mathrm{i} q_{xx} + 2\mathrm{i} V q = 0, \quad V = q(x,t)\overline{q}(-x,t) = V_R(x,t) + \mathrm{i} V_I(x,t) \tag{10}$$

Because of the PT symmetry:

$$V_R(x,t) = V_R(-x,t), \quad V_I(x,t) = -V_I(-x,t)$$

The Lax pair

$$\Phi_x(x,t;\lambda) = M(x,t;\lambda)\Phi(x,t;\lambda) = [-\mathrm{i}\lambda\sigma + Q(x,t)]\Phi(x,t;\lambda),$$

$$\Phi_t(x,t;\lambda) = N(x,t;\lambda)\Phi(x,t;\lambda) = [-2\mathrm{i}\lambda^2\sigma + \widetilde{Q(x,t)}]\Phi(x,t;\lambda)$$

$$\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad Q(x,t) = \begin{pmatrix} 0 & q(x,t) \\ r(x,t) & 0 \end{pmatrix},$$

$$\widetilde{Q(x,t)} = 2Q(x,t)\lambda - \mathrm{i}\sigma Q^2(x,t) + \mathrm{i}\sigma Q_x(x,t), \quad r(x,t) = q^*(-x,t).$$

PT symmetric solution

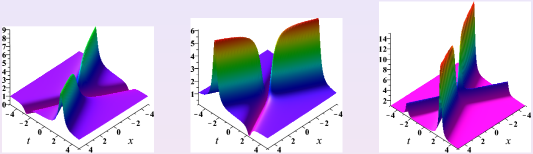
Let $q=\mathrm{e}^{-2\mathrm{i}t}$ and take Darboux transformation

$$q_1=-\frac{F_1}{G_1}\exp(-2\mathrm{i}t)$$

$$\begin{aligned}F_1=&\left(8\,{J_1}^2+8\,{J_2}^2-16\,J_2+8\right)t^2+\left(8\,\mathrm{i}{J_1}^2+8\,\mathrm{i}{J_2}^2-16\,\mathrm{i}J_2+8\,\mathrm{i}+8\,J_1\right)t\\&-\left(2\,{J_1}^2+2\,{J_2}^2-4\,J_2+2\right)x^2+\left(2\,\mathrm{i}{J_1}^2+2\,\mathrm{i}{J_2}^2-2\,\mathrm{i}\right)x\\&+4\,\mathrm{i}J_1-{J_1}^2-{J_2}^2+4\,J_2-1,\\G_1=&\left(8\,{J_1}^2+8\,{J_2}^2-16\,J_2+8\right)t^2+8\,tJ_1-\left(2\,{J_1}^2+2\,{J_2}^2-4\,J_2+2\right)x^2\\&+\left(2\,\mathrm{i}{J_1}^2+2\,\mathrm{i}{J_2}^2-2\,\mathrm{i}\right)x+{J_1}^2+{J_2}^2+1.\end{aligned}$$

If $J_1^2+J_2^2\neq 1$, q_1 is analytic. $J_1,\ J_2$ are phase parameters.

Structure related to phase parameters



(a) (b) (c)

Figure: (Color online) $|q_1|^2$ displays three structures. (a)The dark-bright(DB) structure: $J_1 = 2$ and $J_2 = 3$. (b)The bright-dark(BD) structure: $J_1 = 0.2$ and $J_2 = 0.2$. (c)The bright-bright(BB) structure: $J_1 = 3$ and $J_2 = -1$.

Structure related to phase parameters(Continue)

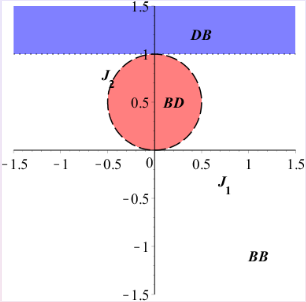


Figure: (Color online) $|q|$ has a DB structure in the blue area, a BD structure in the red area, a BB structure in the complementary area except the boundary lines, $J_2 = 1$, $J_1^2 + (J_2 - \frac{1}{2})^2 = \frac{1}{4}$ and $J_1^2 + J_2^2 = 1$.

Gain/loss profile implied by $V_I(x, t)$

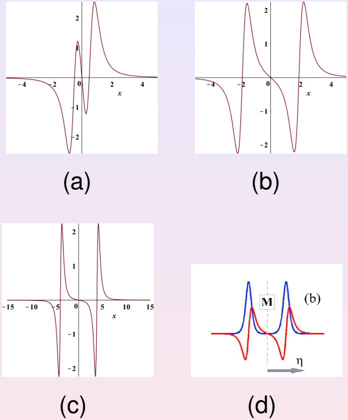


Figure: (Color online) Gain and loss profile. (a) $t = 0$. (b) $t = 1$. (c) $t = 2$. (d)Physical result(red line) (El-Ganainy 2007).

Future work

1. n-dim (nonlocal)NLS and (nonlocal) DNLS hierarchies
2. Bäcklund transformation
3. Bi-Hamiltonian structure
4. Rational solutions

Reference

- M.J. Ablowitz and P.A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, Cambridge University Press, Cambridge, UK, 1991.
- Ablowitz, M.J., Musslimani, Z., *Integrable nonlocal nonlinear Schrödinger equation*, Phys. Rev. Lett. **110** (2013), 064105, 5 pp.
- Drinfel'd, V.G., Sokolov, V.V., *Lie algebras and equations of Korteweg-de Vries type*, Journal of Soviet Mathematics **30(2)** (1985),1975–2036.
- Terng, C.L., Uhlenbeck, K., *The $n \times n$ KdV hierarchy*, J. Fixed Point Theory Appl. **10(1)** (2011), 37–61.
- El-Ganainy R., Makris K., Christodoulides D., Musslimani Z.H., *Theory of coupled optical PT-symmetric structures*, Opt. Lett., **32** (2007), 2632–2634.

Thank You!